1 Notation

Throughout, $G$ denotes a connected graph with vertex set $V$ and edge set $E$, and $\mathbb{R}^E$ and $\mathbb{Z}^E$ denote respectively the set of all real-valued and integer-valued vectors whose coordinates are indexed by the edges of $G$.

For a vector $x$ in $\mathbb{R}^E$ and a set $F$ of edges, $x(F) = \sum_{e \in F} x(e)$.

$M$ denotes the set of all perfect matchings of $G$.

$\chi^M$ denotes the incidence vector of $M$.

Cuts: A cut of $G$ is a subset of $E$ that is the coboundary $\partial(S)$ of some subset $S$ of $V$. For a cut $C := \partial(S)$, $S$ and $\overline{S}$ are the shores of $C$. A cut is trivial if one of its shores is a singleton. A cut is odd if both its shores have odd cardinality.

Note: This is the notation used in Bondy and Murty’s book “Graph Theory (2008)” [1]. Most optimizers use $\delta(S)$, and we ourselves used $\nabla(S)$ instead of $\partial(S)$ in some of our papers [3]-[8].

2 The Perfect Matching Polytope

The Perfect Matching Polytope ($\mathcal{P}_{\text{poly}}(G)$) is the convex hull of $\{\chi^M : M \in \mathcal{M}\}$.

**Theorem 2.1 (Edmonds, 1965 [10])**

A vector $x$ in $\mathbb{R}^E$ belongs to the perfect matching polytope $\mathcal{P}_{\text{poly}}(G)$ of a graph $G$ if and only if it satisfies the following system of linear inequalities:

\[
\begin{align*}
\mathbf{x} & \geq 0 & \text{(nonnegativity)} \\
\mathbf{x}(\partial(v)) &= 1 & \text{for all } v \in V & \text{(degree constraints)} \\
\mathbf{x}(\partial(S)) & \geq 1 & \text{for all odd } S \subset V & \text{(odd set constraints)}
\end{align*}
\]

When $G$ is bipartite, the first two conditions imply the third. This is in general not true, see Figure 1.

**Exercise 2.2** Prove that for bipartite graphs the nonnegativity and the degree constraints imply the odd set constraints.

In the case of the Petersen graph, instead of $2^8$ odd set constraints, only six constraints are necessary, the six cuts whose shores are pentagons (Figure 2).

**Exercise 2.3** Prove that in the case of the Petersen graph, the six odd constraints are necessary.

We now try to identify the graphs that need odd set constraints. We study this question in the context of matching covered graphs.
Figure 1: A nonnegative 1-regular vector that is not in the polytope

Figure 2: The Petersen graph needs only six odd set constraints

3 Matching Covered Graphs

Theorem 3.1 (Tutte, 1947 [20])
A graph $G$ has a perfect matching if and only if $|\mathcal{O}(G - S)| \leq |S|$ for all $S \subset V$, where $\mathcal{O}(G - S)$ denotes the set of odd components of $G - S$.

**Barriers**: In a graph with a perfect matching, a barrier is a subset $S$ of $V$ s.t. $|\mathcal{O}(G - S)| = |S|$.

**Admissible edges**: An edge $e$ of $G$ is admissible if $e \in M$ for some $M \in \mathcal{M}$.

**Exercise 3.2** In a graph with a perfect matching, show that an edge $e$ is admissible iff there is no barrier that contains both ends of $e$.

A Matching Covered Graph is a connected graph on two or more vertices in which every edge is admissible. We restrict our attention to matching covered graphs.

**Exercise 3.3** Show that every 2-connected cubic graph is matching covered.
Figure 3 shows several examples of cubic matching covered graphs.

(a) $K_4$
(b) $C_6$
(c) the Petersen graph
(d) $K_{3,3} = M_6$
(e) $M_8$
(f) the cube = $B_8$

Figure 3: Illustrious cubic matching covered graphs

Critical: A graph $G$ is critical if $G - v$ has a perfect matching for each $v \in V$.

Exercise 3.4 Let $G$ be a graph and let $S$ be a maximal subset of $V$ such that $|\mathcal{O}(G - S)| \geq |S|$. Show that each component of $G - S$ is odd and critical.

Bicritical graphs: A graph $G$ is bicritical if $G - \{u, v\}$ has a perfect matching for any $u, v \in V$. A bicritical graph is a matching covered graph in which every singleton is a maximal barrier.

4 Separating Cuts and Tight Cuts

$C$-contractions: Let $C := \partial(X)$ be a cut of a matching covered graph $G$, where $|X|$ is odd. We denote the graph obtained by shrinking $X$ to a single vertex $x$ by $G/\{X \rightarrow x\}$ and, similarly, the graph obtained from $G$ by shrinking $\overline{X}$ to a single vertex $\overline{x}$ by $G/\{\overline{X} \rightarrow \overline{x}\}$. The two graphs $G/\{X \rightarrow x\}$ and $G/\{\overline{X} \rightarrow \overline{x}\}$ are the $C$-contractions of $G$ (Figure 4).

Figure 4: A $C$-contraction
Separating Cuts: A cut $C$ of $G$ is separating if both $C$-contractions are also matching covered.

**Exercise 4.1** Show that cut $C$ of a matching covered graph $G$ is separating iff, for any $e \in E$, there is a perfect matching $M_e$ such that $e \in M_e$ and $|C \cap M_e| = 1$.

**Tight Cuts**: A cut $C$ of $G$ is tight if $|C \cap M| = 1$ for all $M \in \mathcal{M}$.

Figure 5 shows several examples of tight cuts.

![Figure 5: Tight Cuts](image)

Not every separating cut is tight. For example, the cut $C$ in Figure 1 is a separating cut, but it is not tight! However

**Proposition 4.2**
Every tight cut is a separating cut.

**Exercise 4.3** Let $C := \partial(X)$ be a separating cut in a matching covered graph $G$ that is not tight. In this case, show that both shores of $C$ induce graphs that are not bipartite.

**Exercise 4.4** Deduce from the above exercise that in a bipartite matching covered graph, every separating cut is a tight cut.

**Solid matching covered graphs** A matching covered graph is solid if every separating cut is tight.

**Odd intercyclic graphs** A graph is odd intercyclic if every pair of distinct odd cycles shares at least one vertex.

**Exercise 4.5** Prove that every odd intercyclic matching covered graph is solid.

The following exercise provides a simple characterization of tight cuts in bipartite graphs. If $X$ is an odd subset of the vertex set of a bipartite matching covered graph $G$ with bipartition $(A, B)$, clearly, one of $|X \cap A|$ and $|X \cap B|$ is larger than the other; the larger of the two sets is called the majority part and is denoted by $X_+$, and the smaller is called the minority part and is denoted by $X_-$. (Similarly, the majority and minority parts of $\overline{X} = V \setminus X$ are $\overline{X}_+$ and $\overline{X}_-$, respectively.)

**Exercise 4.6** Let $\partial(X)$ be a tight cut in a bipartite matching covered graph $G[A, B]$. Show that

(i) $|X_+| = |X_-| + 1$, and $|\overline{X}_+| = |\overline{X}_-| + 1$, and
(ii) all edges in the cut $\partial(X)$ have one end in $X_+$ and one end in $\overline{X}_+$. 

Exercise 4.7 Using Exercise 4.3, deduce the following:

(i) Each odd prism (of order $4k + 2$, $k \geq 1$) has precisely one separating cut that is not tight

(ii) The Petersen graph has precisely six separating cuts that are not tight.

4.1 Regular vectors

Let $G$ be a matching covered graph, and let $C$ be a tight cut of $G$. Since every perfect matching of $G$ has, by definition, exactly one edge in $C$, it follows that

$$x(C) = 1, \text{ for every vector } x \text{ in } \mathcal{P}(G) \quad (1)$$

Thus, for each tight cut $C$, the equation $x(C) = 1$ defines a hyperplane in which the polytope $\mathcal{P}(G)$ sits.

For a real number $r$, we define a vector $x$ in $\mathbb{R}^E$ to be $r$-regular if

$$x(C) = r, \text{ for every tight cut } C \text{ of } G \quad (2)$$

By the observation made above, it follows that all vectors in $\mathcal{P}(G)$ are nonnegative 1-regular vectors. However, as noted earlier, not every nonnegative 1-regular vector need be a member of $\mathcal{P}(G)$ (see Figure 1).

We shall show later that the class of graphs whose perfect matching polytopes consist of all nonnegative 1-regular graphs is precisely the class of solid graphs.

Problem 4.8 (Unsolved) Is there a polynomial time algorithm to determine whether a given matching covered graph is solid?

We do have a polynomial time algorithm to recognize solid planar graphs.

Exercise 4.9 Using Exercise 4.6 deduce that if $G$ is bipartite then every vector $x$ in $\mathbb{R}^E$ which satisfies the degree condition in the description of the matching polytope is 1-regular. (Thus, in case of bipartite graphs, if the condition $x(C) = 1$ holds for all trivial tight cuts, then it holds for all tight cuts.)

Exercise 4.10 For each graph in Figure 5, give examples of vectors $x$ that are nonnegative and satisfy the degree constraints but are not 1-regular (and consequently do not lie in $\mathcal{P}(G)$).

5 Bricks and Braces

Barrier cuts: For any barrier $B$ and any odd component $K$ of $G - B$, $\partial(V(K))$ is a tight cut. Such cuts are called barrier cuts. (See Figure 5(a))

2-separation cuts: For any 2-separation $\{u, v\}$ of $G$ and any even component $L$ of $G - \{u, v\}$, $\partial(V(L) \cup \{u\})$ and $\partial(V(L) \cup \{v\})$ are tight cuts. Such cuts are called 2-separation cuts. (See Figure 5(b).)

A graph may have tight cuts that are neither barrier cuts nor 2-separation cuts. (See Figure 5(c).) However:
**Theorem 5.1** (Edmonds, Lovász, Pulleyblank, 1982 [12])

Every graph that has a nontrivial tight cut either has a nontrivial barrier or a 2-separation.

New proofs of this result appear in Szigeti, 2002 [19] and in CLM, 2014 [9].

**Braces:** A brace is a bipartite matching covered graph that has no nontrivial tight cuts. (A bipartite graph $G$ with bipartition $(A, B)$, $|V| \geq 4$, is a brace iff, for any $a_1, a_2 \in A$ and $b_1, b_2 \in B$, the graph $G - \{a_1, a_2, b_1, b_2\}$ has a perfect matching.) The cube and $K_{3,3}$ and all prisms of order $4k$, $k \geq 2$ and all Möbius ladders of order $4k + 2$, $k \geq 1$, are braces.

**Bricks:** A brick is a nonbipartite matching covered graph that has no nontrivial tight cuts. (A graph $G$ is a brick iff it is bicritical and 3-connected. A proof of this requires Theorem 5.1. There is a polynomial-time algorithm for deciding whether or not a given $G$ is a brick.) The graphs $K_4$, $C_6$, all Möbius ladders of order $4k$, $k \geq 1$, and all prisms of order $4k + 2$, $k \geq 1$, are bricks.

**Tight Cut Decomposition:** By repeatedly taking contractions with respect to nontrivial tight cuts, any graph may be decomposed into bricks and braces. For example, up to multiple edges, the tight cut decompositions of the graphs in Figure 5 produce, respectively, (a) two $K_4$'s and $K_{3,3}$, (b) two $K_4$'s, and (c) two $K_4$'s and $K_{3,3}$.

**Theorem 5.2** (Lovász, 1987 [15])

Any two tight cut decompositions of a matching covered graph $G$ yield the same list of bricks and braces (except possibly for multiplicities of edges).

Figure 6 depicts an example of the uniqueness of tight cut decompositions, up to multiple edges.

![Figure 6: An example of the uniqueness of tight cut decompositions](image)

**Exercise 5.3** Find all the tight cut decompositions of the graph in Figure 5(c).

**The number of bricks:** The number of bricks resulting from a tight cut decomposition of $G$, denoted by $b(G)$, is an invariant of $G$. A graph $G$ is a near-brick if $b(G) = 1$. 
Theorem 5.4
Let \( C \) be a tight cut of \( G \) and let \( G_1 \) and \( G_2 \) be the two \( C \)-contractions of \( G \). A vector \( x \) in \( \mathbb{R}^E \) belongs to \( \operatorname{Poly}(G) \) iff the restrictions of \( x \) to \( E(G_1) \) and \( E(G_2) \) belong, respectively, to \( \operatorname{Poly}(G_1) \) and \( \operatorname{Poly}(G_2) \).

Thus, to check if a vector \( x \) is \( \operatorname{Poly}(G) \), it suffices to check whether or not the restrictions of \( x \) to the edge sets of the bricks and braces are in the perfect matching polytopes of those graphs. For this reason, in seeking an answer to Problem 4.8, we may restrict our attention to bricks.

6 Solid Bricks

A matching covered graph \( G \) is solid if it has no separating cuts other than tight cuts. Since a brick has no tight cuts other than the trivial cuts, it follows that a brick is solid if and only if it has no nontrivial separating cuts.

We introduced and made use of solid bricks in proving a conjecture of Lovász ([4] and [5]). (This will be described later on.) One of the notions that played a useful role in that work was a relation defined on the set of cuts of a graph.

A precedence relation on cuts: Let \( C \) and \( D \) be two cuts of a graph \( G \). Cut \( D \) precedes cut \( C \) (written as \( D \preceq C \)) if \( |M \cap D| \leq |M \cap C| \) for each perfect matching \( M \) of \( G \).

Example 6.1 Let \( G \) be a brick and let \( C := \partial(X) \) be a nontrivial odd cut of \( G \). If \( C \) is not a separating cut, then one of the two \( C \)-contractions is not matching covered. Suppose that \( G_1 := G/\{X \to x\} \) is not matching covered. Then, either (i) \( G_1 \) has no perfect matching, or (ii) \( G_1 \) has a perfect matching, but it has an edge that is inadmissible. In the first case, there exists a subset \( S \) of \( V(G_1) \) such that \( |O(G_1 - S)| > |S| \), and in the second case, there is a barrier \( S \) of \( G_1 \) that contains both ends of some edge \( e \) of \( G \). Since \( G \) is a brick there is no subset \( S \) of \( V(G) \) with either of these properties. In both alternatives, the contraction vertex \( x \) lies in \( S \) (Figure 7).

![Figure 7: The case \( |O(G - S)| = |S| \)](image)

Suppose that \( K \) is an odd component of \( G_1 - S \) and let \( D := \partial(V(K)) \). In case (i), \( |D \cap M| < |C \cap M| \) for every perfect matching \( M \) of \( G \). In case (ii), \( |D \cap M| \leq |C \cap M| \) for every perfect matching \( M \) of \( G \), with equality only if \( e \) does not lie in \( M \). It follows that, in either case, \( D \) strictly precedes \( C \).

If a brick \( G \) is nonsolid then, by definition, it has nontrivial a separating cut, say \( C \), and the two \( C \)-contractions \( G_1 \) and \( G_2 \) of \( G \) are matching covered. But, in general, \( G_1 \) and \( G_2 \) need not be bricks or even near-bricks. For the purpose of applying induction to prove Lovász’s conjecture, it was necessary for us to find a separating cut \( C \) such that both \( G_1 \) and \( G_2 \) are near-bricks. We called such a separating cut a robust cut and proved the following theorem.
Theorem 6.2
Every nonsolid brick has a robust cut.

Given any separating cut \( C \) of a brick \( G \), we showed that either \( C \) is a robust cut or there is a separating cut \( D \) that precedes \( C \) strictly. Thus, any separating cut that is minimal with respect to the precedence relation is a robust cut. We also proved the following generalization of the above theorem.

Theorem 6.3
In any nonsolid brick \( G \) there are two separating cuts \( \partial(X) \) and \( \partial(Y) \), \( X \cap Y = \emptyset \), such that \( G/X \) and \( G/Y \) are bricks and the graph obtained from \( G \) by shrinking \( X \) to \( x \) and \( Y \) to \( y \) is bipartite.

7 A Solution to the Problem for Bricks

Theorem 7.1
For a brick \( G \), \( \text{Poly}(G) \) consists of all nonnegative 1-regular vectors if and only if \( G \) is solid.

Proof: Firstly suppose that \( G \) is not solid. We wish to show that there is some nonnegative 1-regular vector in \( \mathbb{R}^E \) that does not belong to \( \text{Poly}(G) \). Since \( G \) is nonsolid, it has a nontrivial separating cut \( C \). Let \( M_0 \) be a perfect matching of \( G \) such that \( |M_0 \cap C| > 1 \). (Such a perfect matching must exist; otherwise \( C \) would be tight.) Also, since \( C \) is separating, for every edge \( e \) of \( G \), there is a perfect matching \( M_e \) of \( G \) such that \( M_e \cap C = \{e\} \). Now let

\[
\mathbf{x} := \frac{1}{|M_0| - 1} \left( \left( \sum_{e \in M_0} x_e \right) - x_{M_0} \right)
\]

Clearly the vector \( \mathbf{x} \) is nonnegative, 1-regular with \( x(C) < 1 \).

Conversely, suppose that \( G \) is solid. We wish to prove that every nonnegative 1-regular vector in \( \mathbb{R}^E \) belongs to \( \text{Poly}(G) \). Assume to the contrary that there is a nonnegative 1-regular vector \( \mathbf{x} \) that does not belong to \( \text{Poly}(G) \). Then, by Theorem 2.1 there must exist odd cuts \( C \) with \( x(C) < 1 \). Let \( \mathcal{C} \) denote the set of all cuts \( C \) for which \( x(C) < 1 \) and let \( D := \partial(Y) \) be a cut in \( \mathcal{C} \) that is minimal with respect to the precedence relation \( \preceq \) defined in the previous section. We shall obtain a contradiction by showing that \( D \) is a separating cut.

Consider the \( D \)-contraction \( G_1 := G/Y \). We wish to show that \( G_1 \) is matching covered. If it is not, then either there is a subset \( S \) of \( V(G_1) \) such that either (i) \( |S| > |S| \) or (ii) \( |\partial(G_1 - S)| = |S| \), but there is an edge \( e \) of \( G_1 \) with both its ends in \( S \). In either case, there must be an odd component \( K \) of \( G_1 - S \) for which \( x(D') < 1 \), where \( D' := \partial(V(K)) \). Such a component \( K \) is clearly nontrivial. One may verify that \( D' \) strictly precedes \( D \) (see Example 6.1), contradicting the choice of \( D \). Therefore \( G_1 \) is matching covered. Similarly, \( G_2 := G/S \) is also matching covered and \( D \) is a separating cut. A contradiction.

Using Theorem 6.3 it is possible to establish the following characterization of nonsolid bricks.

Theorem 7.2
A brick \( G \) has a nontrivial separating cut iff it there exists two disjoint subsets \( X \) and \( Y \) of \( V \) such that (i) \( G[X] \) and \( G[Y] \) are nontrivial critical graphs, and (ii) \( G - (X \cup Y) \) has a perfect matching.

The above theorem is a variant of the following attractive theorem.
Theorem 7.3 (Reed and Wakabayashi)
A brick $G$ has a nontrivial separating cut iff it has two odd circuits $C_1$ and $C_2$ such that the graph $G - (V(C_1) \cup V(C_2))$ has a perfect matching.

8 Examples of Solid Bricks

A graph is odd-intercyclic if any two odd circuits of $G$ have at least one vertex in common. By Theorem 7.3, all odd-intercyclic bricks are solid. (This can be proved by elementary arguments quite easily.) Odd wheels and Möbius ladders described below are examples of odd-intercyclic solid bricks.

Odd Wheels: The wheel of order $n \geq 3$, denoted by $W_n$, is obtained by adjoining a vertex $h$ to a circuit $R$ of length $n$ and joining it to each vertex of $R$; $h$ and $R$ are referred to as the hub and the rim of $W_n$, respectively. (The wheel $W_3$ of order three is isomorphic to $K_4$; any one of its four vertices may be regarded as its hub.) A wheel is odd or even according to the parity of its order. It is easy to show that every odd wheel is an odd-intercyclic brick.

Möbius ladder: The ladder $L_{2n}$, $n \geq 2$, is obtained from two disjoint paths $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ by adding the edges $x_iy_i$, $1 \leq i \leq n$. The Möbius ladder $M_{2n}$, $n \geq 2$, is obtained from $L_{2n}$ by joining $x_1$ to $y_n$ and $y_1$ to $x_n$. This graph is Hamiltonian and is isomorphic to the cubic graph obtained from the circuit $(0, 1, ..., 2n - 1)$ by joining each vertex $i$ to the vertex $i + n \pmod{2n}$. Figure 8 depicts the Möbius ladder $M_8$.

![Figure 8: The Möbius ladder $M_8$](image)

It can be shown that, for any odd integer $n \geq 3$, $M_{2n}$ is a brace, and for any even integer $n \geq 2$, $M_{2n}$ is an odd-intercyclic brick.

8.1 An infinite family of solid bricks that are not odd-intercyclic

Let $n \geq 3$ be an odd integer. Consider the brace $M_{2n}$, which is the Möbius ladder on $2n$ vertices. Obtain the cubic graph $H$ from $M_{2n}$ by deleting the vertex $n$, adding three new vertices $u$, $v$ and $w$, and joining $u$ to $n - 1$, $v$ to $0$, $w$ to $n + 1$ and $u$, $v$ and $w$ to each other. (Thus $H$ is obtained by splicing $M_{2n}$ and $K_4$.) Now obtain the graph $S_{2n+2}$ from $H$ by joining $1$ and $2n - 1$. See Figure 9. It can be shown that $S_{2n+2}$ is a solid brick for every odd integer $n \geq 3$. In fact, Murty devised this family as a generalization of the first graph in the family, $S_8$, which he discovered, and is the smallest non-odd-intercyclic solid matching covered graph.

Exercise 8.1 Show that the graph $S_8$ (Figure 9, $n = 3$) is a non-odd-intercyclic solid brick.
There is a characterization of odd-intercyclic graphs:

**Theorem 8.2 (Kawarabayashi and Ozeki [13])**

Let \( G \) be an internally 4-connected graph. Then \( G \) is odd-intercyclic if and only if \( G \) satisfies one of the following:

1. either \( G - x \) is bipartite for some vertex \( x \in V \); or
2. \( G \) has a triangle \( T \) such that \( G - T \) is bipartite; or
3. \( |V| \leq 5 \); or
4. \( G \) can be embedded into the projective plane so that every face boundary has even length.

One general result that we have been able to prove is that odd wheels are the only simple planar solid bricks [8].

Solid bricks behave much as bipartite graphs. We have proved that every cubic solid brick has a 3-edge-colouring [17].

9 **The Matching Lattice and its Bases**

9.1 **The Linear Space**

\( \mathcal{L}\text{in}(G) \) denotes the set of linear combinations of characteristic vectors of perfect matchings of a matching covered graph \( G \). That is,

\[
\mathcal{L}\text{in}(G) := \sum_{M \in \mathcal{M}} \alpha(M)\chi^M,
\]
where each coefficient $\alpha(M)$ is a real number. Note that if $x = \sum \alpha(M)\chi^M$ is any vector in $\mathcal{L}(G)$, and $C$ is any tight cut of $G$, then

$$x(C) = \sum_{M \in \mathcal{M}} \alpha_M \chi^M(C) = \sum_{M \in \mathcal{M}} \alpha_M$$

As this is true for any tight cut $C$, it follows that every vector in $\mathcal{L}(G)$ is regular. Conversely, it can be shown that all regular vectors are in $\mathcal{L}(G)$.

**Theorem 9.1 (Edmonds, Lovász, Pulleyblank [12])**

Let $G$ be a matching covered graph. A vector $x$ in $\mathbb{R}^E$ lies in $\mathcal{L}(G)$ if and only if it is regular. Moreover, the dimension of $\mathcal{L}(G)$ satisfies the formula

$$\dim(\mathcal{L}(G)) = m - n + 2 - b,$$

where $m$, $n$ and $b$ denote respectively the number of edges, vertices and bricks of $G$.

We shall abuse the language and refer to a set $B$ of perfect matchings as a basis, meaning actually the set $\{\chi^M : M \in B\}$ of the corresponding incidence vectors. We shall denote the basis by using boldface, thus, $\mathbf{B}$ denotes the basis consisting of the incidence vectors of matchings in $B$.

**Exercise 9.2** Figure 10 depicts an example of a matching covered graph $G$ and a regular vector $x \in \mathbb{R}^E$. The parameters for graph $G$ are $m = 15$, $n = 10$ and $b = 2$. Thus, $\dim(\mathcal{L}(G)) = 5$. Find a basis of $\mathcal{L}(G)$ consisting of 5 perfect matchings and express $x$ as a linear combination of these five matchings.

**Exercise 9.3** For the graph $G$ of Figure 10 give an example of a nonregular vector $x$ in $\mathbb{R}^E$ such that $x(\partial(v)) = x(\partial(w))$ for any each pair $\{v, w\}$ of vertices of $G$.

**Exercise 9.4**

(i) Let $r$ be any real number. If $x$ and $y$ are any two $r$-regular vectors in $\mathbb{R}^E$, then show that the vector $\alpha x + \beta y$ is also an $r$-regular vector, for any $\alpha$ and $\beta$ such that $\alpha + \beta = 1$.

(ii) By Edmonds’ theorem all non-negative 1-regular vectors are in $\mathcal{L}(G)$. Suppose that $y$ is a 1-regular vector that is not in $\mathcal{P}(G)$. Express $y$ as a linear combination of two vectors in $\mathcal{P}(G)$ and thereby show that $y$ is also $\mathcal{L}(G)$. (Hint: Take $x$ to be any
strictly positive vector in \( \text{Poly}(G) \) (such a vector must exist because every edge of \( G \) is in a perfect matching). Then, by the first part, the vector \( z := (1 - \epsilon)x + \epsilon y \) is 1-regular for any real number \( \epsilon \). Clearly, for small enough values of \( \epsilon \), the vector \( z \) is also non-negative, and hence is in \( \text{Poly}(G) \).

(iii) Now suppose \( r \neq 1 \), and let \( y \) be an \( r \)-regular vector. Show that \( y \) is in \( \text{Lin}(G) \) by showing that, for any 1-regular vector \( x \), the vector \( \frac{1}{1-r}(x - y) \) is 1-regular.

**Exercise 9.5**

(i) Let \( T \) denote the matrix whose rows are the incidence vectors of all tight cuts of a matching covered graph \( G \). For any fixed real number \( r \), show that the set of all \( r \)-regular vectors of \( G \) is the set of solutions to the system \( Tx = r \) of linear equations, where \( r \) is a column vector each of whose entries is \( r \).

(Thus, the set of all regular vectors is the set of solutions to the system \( Tx = r \), where \( r \) is treated as a variable).

When \( G \) is either a brace or a brick, all tight cuts are trivial. Thus, in this case, \( T \) is the same as the incidence matrix \( A \) of \( G \). Consequently, \( \text{Lin} \) is the set of solutions to the system \( Ax - r = 0 \) of homogeneous linear equations in \( m + 1 \) variables, and its dimension is \( (m + 1) - \text{rank}(A) \).

(ii) Show that the rank of \( A \) is \( n - 1 \) when \( G \) is bipartite, and is \( n \), when \( G \) non-bipartite, and deduce the validity of the dimension formula for braces and bricks.

### 9.2 Robust Cuts and Regularity

The regularity of a vector \( x \in \mathbb{R}^E \) is obviously necessary for \( x \) to be in \( \text{Lin}(G) \). Let us now consider matching covered graphs \( G \) that are either bipartite or near-bricks. That is, \( b(G) \leq 1 \). Let \( x \) be a vector in \( \mathbb{R}^E \) and \( r \in \mathbb{R} \) such that \( x(\partial(v)) = r \) for any each vertex \( v \) of \( G \). For any tight cut \( C \) of \( G \), the inequality \( b(G) \leq 1 \) implies that one \( C \)-contraction of \( G \) is bipartite, therefore \( x(C) = r \).

In other words, whenever \( b(G) \leq 1 \) the “regularity on the vertices” implies regularity over all tight cuts. We then have the following very important observation.

**Lemma 9.6**

Let \( G \) be a brick, let \( C \) be a robust cut of \( G \), let \( x \in \mathbb{R}^E \) be an \( r \)-regular vector, \( r \in \mathbb{R} \). If \( x(C) = r \) then the restriction of \( x \) to each \( C \)-contraction of \( G \) is \( r \)-regular.

### 9.3 The Matching Lattice

\( \text{Lat}(G) \) denotes the set of integral linear combinations of characteristic vectors of perfect matchings of a matching covered graph \( G \). That is,

\[
\text{Lat}(G) := \sum_{M \in \mathcal{M}} \alpha(M) \chi^M,
\]

where each coefficient \( \alpha(M) \) is an integer. Certainly,

\[
\text{Lat}(G) \subseteq \mathbb{Z}^E \cap \text{Lin}(G).
\]
If equality holds then the characterizations and results for $\mathcal{L}\text{in}(G)$ could be easily adapted to $\mathcal{L}\text{at}(G)$. However, equality does not hold: let $P$ be the Petersen graph. It has precisely six perfect matchings, and each edge lies in precisely two perfect matchings. Thus the vector $x \in \mathbb{R}^E$ of all 1’s is equal to $\sum_{M \in \mathcal{M}} \frac{1}{2} \chi^M$. So, $x$ lies in $\mathcal{L}\text{in}(P)$. By Theorem 9.1, $\dim(\mathcal{L}\text{in}(P)) = 6$, therefore the linear combination is unique. We deduce that $x$ does not lie in $\mathcal{L}\text{at}(P)$.

We now state a fundamental result. A Petersen brick is a brick whose underlying simple graph is the Petersen graph.

**Theorem 9.7 (Lovász [14, 15])**

Let $G$ be a matching covered graph. The dimension of $\mathcal{L}\text{at}(G)$ satisfies the equality

$$\dim(\mathcal{L}\text{at}(G)) = \dim(\mathcal{L}\text{in}(G)) = m - n + 2 - b.$$  

Moreover, if no brick of $G$ is a Petersen brick then a vector $x \in \mathbb{Z}^E$ lies in $\mathcal{L}\text{at}(G)$ if and only if $x$ is regular.

### 9.4 Two Important Conjectures

Let $G$ be a matching covered graph. An edge $e$ of $G$ is removable if $G - e$ is also matching covered.

**Theorem 9.8 (Lovász)**

Every brick distinct from $K_4$ and $C_6$ has a removable edge.

An edge $e$ of $G$ is $b$-invariant if $e$ is removable and $b(G - e) = b(G)$. Figure 11 shows an edge $e$ that is removable but not $b$-invariant and also a $b$-invariant edge $f$. It is important to notice that the Petersen graph has no $b$-invariant edge.

![Figure 11: Edge $e$ is removable in the brick $G$, but $b(G - e) = 2$; edge $f$ is $b$-invariant](image)

Lovász considered the possibility of a simpler proof of Theorem 9.7 if the following result were true:

**Conjecture 9.9 (Lovász)**

Every brick distinct from $K_4$, $C_6$ and the Petersen graph has a $b$-invariant edge.

Also, Murty posed the following conjecture, which extends a trivial result for $\mathcal{L}\text{in}(G)$ to $\mathcal{L}\text{at}(G)$.

**Conjecture 9.10 (Murty)**

Every matching covered graph $G$ has a basis for $\mathcal{L}\text{at}(G)$ that consists solely of perfect matchings.
In his Ph. D. Thesis, Marcelo Carvalho proved Lovász’ Conjecture. He in fact proved a result that is slightly stronger:

**Theorem 9.11 (Carvalho 1997 [2], CLM 2002 [4, 5])**

Let \( G \) be a brick that is distinct from \( K_4 \) and \( C_6 \). If \( G \) is not a Petersen brick then \( G \) has a \( b \)-invariant edge \( e \) such that the brick of \( G - e \) is not the Petersen brick.

### 9.5 Proof of Theorem 9.7 and Conjecture 9.10

Let us prove Lovász’ Theorem 9.7 and Murty’s Conjecture 9.10 by induction, following seminal ideas used by Seymour [18]. We shall prove the sufficiency of regularity and also the existence of bases for \( \text{Lat}(G) \) consisting of \( m - n + 2 - b \) perfect matchings. In order to reduce to braces and bricks, we use the operation of **merging**.

#### 9.5.1 The Merger Operation

Let \( C \) be a separating cut of a matching covered graph \( G \). We denote by \( \text{Lat}(G, C) \) the subspace of \( \text{Lin}(G) \) spanned by the collection of perfect matchings of \( G \) that contain just one edge in \( C \). In particular, if \( C \) is a tight cut then \( \text{Lat}(G, C) = \text{Lat}(G) \).

Let \( G_1 \) and \( G_2 \) denote the two \( C \)-contractions of \( G \). For \( i = 1, 2 \), let \( B_i \) be a collection of perfect matchings of \( G_i \) such that \( B_i \) is a basis for \( \text{Lat}(G_i) \). For each edge \( e \) in \( C \), let \( B^e_i \) denote the subcollection of \( B_i \) consisting of those perfect matchings that contain edge \( e \), let \( F^e_i \) be a fixed matching in \( B^e_i \). See Figure 12. Let

\[
B^e := \{ M_1 \cup F^e_2 : M_1 \in B^e_1 \} \cup \{ F^e_1 \cup M_2 : M_2 \in B^e_2 \} \quad \text{and let} \quad F^e := F^e_1 \cup F^e_2.
\]

Clearly, the set \( B := \cup_{e \in C} B^e \) is a set of perfect matchings of \( G \). We shall denote it by \( B_1 \vee B_2 \) and refer to it as the **merger** of \( B_1 \) and \( B_2 \).

It follows that \( |B^e| = |B^e_1| + |B^e_2| - 1 \), since \( F^e \) is counted twice in the sum \( |B^e_1| + |B^e_2| \). Consequently,

\[
|B_1 \vee B_2| = |B_1| + |B_2| - |C|.
\] (3)
Exercise 9.12 Prove that $B^e$ is linearly independent. Conclude that $B_1 \vee B_2$ is linearly independent.

Lemma 9.13
Assume that, for $i = 1, 2$, the restriction $x_i$ of a vector $x \in \mathbb{Z}^E$ to $G_i$ lies in $\text{Lat}(G_i)$. Then, $x$ lies in $\text{Lat}(G, C)$ and is an integral linear combination of matchings in $B_1 \vee B_2$.

Proof: By hypothesis, $B_i$ is a basis of $\text{Lat}(G_i)$. Thus, there exist integral coefficients $\alpha(M), M \in B_i$, such that

$$x_i = \sum_{M \in B_i} \alpha(M) M \quad i = 1, 2.$$ 

It follows that

$$x = \sum_{e \in C} \left[ \sum_{M \in B_i^e} \alpha(M)(M \cup F^e_i) + \sum_{M \in B_2^e} \alpha(M)(F^e_1 \cup M) - x(e)F^e \right].$$

Consequently, $x$ lies in $\text{Lat}(G, C)$ and is an integral linear combination of matchings in $B_1 \vee B_2$. \(\square\)

Corollary 9.14
The merger $B_1 \vee B_2$ is a basis for $\text{Lat}(G, C)$.

Proof: Let $x \in \text{Lat}(G, C)$. Then, $x$ is an integral linear combination of perfect matchings of $G$ that contain precisely one edge in $C$. Thus, the restrictions $x_i$ of $x$ to $G_i$ lie in $\text{Lat}(G_i)$. It follows that $B_1 \vee B_2$ spans $x$. This conclusion holds for each $x \in \text{Lat}(G, C)$. \(\square\)

For $i = 1, 2$, let $m_i, n_i$ and $b_i$ denote, respectively, the number of edges, vertices and bricks of $G_i$. By induction, $|B_i| = m_i - n_i + 2 - b_i$. Then,

$$\dim(\text{Lat}(G, C)) = |B_1 \vee B_2| = |B_1| + |B_2| - |C|$$

$$= (m_1 - n_1 + 2 - b_1) + (m_2 - n_2 + 2 - b_2) - |C|$$

$$= (m_1 + m_2) - |C| - (n_1 + n_2 - 2) + 2 - (b_1 + b_2)$$

$$= m - n + 2 - (b_1 + b_2). \tag{4}$$

9.5.2 Reduction to Bricks and Braces
Let $C$ be a nontrivial tight cut of $G$, let $G_1$ and $G_2$ denote the two $C$-contractions of $G$. From (4), we deduce that $\text{Lat}(G)$ has a basis consisting of $m - n + 2 - b$ perfect matchings.

Assume that a vector $x \in \mathbb{Z}^E$ is $r$-regular, $r \in \mathbb{Z}$. Every tight cut of $G_i$ is also a tight cut of $G$. Thus, the restriction $x_i$ of $x$ to $G_i$ is $r$-regular. Assume also that no brick of $G_i$ is a Petersen brick.

By induction, $x_i$ lies in $\text{Lat}(G_i)$. By Lemma 9.13, $x$ lies in $\text{Lat}(G)$.

Using induction and the merger operation, we have reduced the proof of Theorem 9.7 to the case where $G$ is a brace or a brick.

9.5.3 Reduction to Braces and Solid Bricks
We now use the merger operation and the fact that a nonsolid brick has a robust cut to reduce the problem to braces and solid bricks.

Let $G$ be a nonsolid brick. If $G$ is the Petersen graph then its set of six perfect matchings are linearly independent. Moreover, $m - n + 1 = 6$, the assertion holds.
If \( G \) is a Petersen brick, we may “coalesce” parallel edges and use the previous case to prove that it also has a basis consisting of \( m - 9 \) perfect matchings.

We may thus assume that \( G \) is not a Petersen brick. The following result is a fundamental property of bricks which are not Petersen bricks.

**Theorem 9.15** (Carvalho 1997 [2], CLM 2002 [4, 5])

Let \( G \) be a brick that is not a Petersen brick. Then, \( G \) has a robust cut \( C \) such that (i) the bricks of the \( C \)-contractions of \( G \) are not Petersen bricks and (ii) \( G \) has a perfect matching \( M_0 \) such that \( |M_0 \cap C| = 3 \).

Figure 13 illustrates a brick \( G \) and a robust cut \( C \) as in the statement of Theorem 9.15.

Let \( x \in \mathbb{Z}^E \) be an \( r \)-regular vector. Let

\[
\beta := \frac{x(C) - r}{|M_0 \cap C| - 1}
\]

and let \( y := x - \beta \cdot M_0 \). \hspace{1cm} (5)

Clearly, \( x(C) \) and \( r \) have the same parity. It follows that \( \beta \) is integral. Moreover, \( y(C) + \beta = r \), whence \( y \) is \( s \)-regular, where \( s = r - \beta \). Also, \( y(C) = s \). As \( G_i \) is a near-brick, the restriction \( y_i \) of \( y \) to \( G_i \) is \( s \)-regular. By induction, \( y_i \) lies in \( \mathcal{L}(G_i) \). By Lemma 9.13, \( y \) is spanned by \( B_1 \vee B_2 \). Thus, \( B \) is linearly independent.

Let \( x \in \mathbb{Z}^E \) be an \( r \)-regular vector. Let

\[
\beta := \frac{x(C) - r}{|M_0 \cap C| - 1}
\]

and let \( y := x - \beta \cdot M_0 \). \hspace{1cm} (5)

Clearly, \( x(C) \) and \( r \) have the same parity. It follows that \( \beta \) is integral. Moreover, \( y(C) + \beta = r \), whence \( y \) is \( s \)-regular, where \( s = r - \beta \). Also, \( y(C) = s \). As \( G_i \) is a near-brick, the restriction \( y_i \) of \( y \) to \( G_i \) is \( s \)-regular. By induction, \( y_i \) lies in \( \mathcal{L}(G_i) \). By Lemma 9.13, \( y \) is spanned by \( B_1 \vee B_2 \). Thus, \( x \) is spanned by \( B \) and lies in \( \mathcal{L}(G) \). We conclude that \( B \) is a basis of \( \mathcal{L}(G) \) and spans every regular vector in \( \mathbb{Z}^E \).

Again, by the use of induction and the merger operation we have advanced more, now we are left with the case in which \( G \) is either a brace or a solid brick.
9.5.4 Braces and Solid Bricks

**Theorem 9.16 (Lovász [14, 16])**

Every brace distinct from $K_2$ and $C_4$ has a removable edge. Every brick distinct from $K_4$ and $C_6$ has a removable edge.

**Theorem 9.17 (Carvalho 1997 [2], CLM 2002 [4])**

Let $e$ be a removable edge of a solid brick $G$. Then, $e$ is $b$-invariant and $G - e$ is solid.

The assertion of Theorem 9.7 holds trivially for $K_2$, $C_4$ and $K_4$. The brick $C_6$ is not solid. For any other brace or solid brick $G$, let $e$ be a $b$-invariant edge of $G$. Let $M_e$ be any perfect matching of $G$ that contains edge $e$. By induction, $\text{Lat}(G - e)$ has a basis consisting of $m - n + 1 - b$ perfect matchings. Add $M_e$ to that basis, thereby obtaining a set $B$. Clearly, $B$ is linearly independent, because $B \setminus \{M_e\}$ is linearly independent and none of its perfect matchings contains edge $e$. For any regular vector $x$ in $\mathbb{Z}^E$, the restriction $y$ of vector $x - x(e) \cdot M_e$ to $E(G - e)$ is regular. Moreover, $G - e$ is a solid matching covered graph, therefore its brick is not a Petersen brick. By induction hypothesis, $y$ lies in $\text{Lat}(G - e)$. We conclude that $x$ lies in $\text{Lat}(G)$. Indeed, $B$ spans every regular vector in $\mathbb{Z}^E$. Thus, $B$ is a basis of $G$ consisting of $m - n + 1$ perfect matchings and every regular vector in $\mathbb{Z}^E$ lies in $\text{Lat}(G)$. The proof of Theorem 9.7 is complete.

**Exercise 9.18** Find a basis for the matching lattice of the graphs depicted in Figures 10, 11 and 13.

10 Algorithmic Proof

The proof we gave so far may not yield a polynomial algorithm for determining a basis for $\text{Lat}(G)$. The reason is that we do not know how to find separating cuts efficiently (Problem 4.8).

In his Ph. D. thesis [2], (see also [4, 5]) Marcelo Carvalho proved the following fundamental result, fully solving Conjecture 9.9. We denote by $(b+p)(G)$ the invariant consisting of the number of bricks of $G$, where the Petersen bricks are counted twice.

**Theorem 10.1 (Carvalho 1997 [2], CLM 2002 [4, 5])**

Every brick $G$ distinct from $K_4$ and $C_6$ has a removable edge $e$ such that $(b+p)(G-e) = (b+p)(G)$.

With this result, we get in polynomial time a basis for $\text{Lat}(G)$ for every matching covered graph $G$, as (i) Theorem 5.1 indicates how to find tight cuts in polynomial time, (ii) there are polynomial time algorithms for determining whether a graph has a perfect matching [11].

11 Open Problems

**Problem 11.1** Is solidity of a brick in P? Is it in NP?

We know that solidity of bricks is in co-NP. Kawarabayashi and Ozeki [13] have a characterization of odd intercyclic graphs that gives a polynomial algorithm for the recognition of odd intercyclic graphs.

**Conjecture 11.2 (Murty)**

There exists a constant $c$ such that every simple solid graph of order $2n \geq c$ has at most $n^2$ edges and this limit is only attained by $K_{n,n}$.
Conjecture 11.3 (Murty)  
There exists a constant $k$ such that every simple solid brick of order $n$ has at most $kn$ edges.

Conjecture 11.4 (Murty)  
Every cubic solid brick is odd intercyclic.

References


Erratum

Exercise 9.4

(i) Let \( r \) be any real number. If \( x \) and \( y \) are any two \( r \)-regular vectors in \( \mathbb{R}^E \), then show that the vector \( \alpha x + \beta y \) is also an \( r \)-regular vector, for any \( \alpha \) and \( \beta \) such that \( \alpha + \beta = 1 \).

(ii) Every vector in \( \text{Poly}(G) \) is clearly in \( \text{Lin}(G) \). Suppose that \( y \) is a 1-regular vector that is not in \( \text{Poly}(G) \). Express \( y \) as a linear combination of two vectors in \( \text{Poly}(G) \) and thereby show that \( y \) is also in \( \text{Lin}(G) \).

Hint: Let \( x = \sum_{M \in \mathcal{M}} \alpha(M) \chi^M \) be a vector in \( \text{Poly}(G) \) in which each \( \alpha(M) \) is strictly positive. Then, \( x \) is 1-regular and strictly positive. Furthermore, for each odd cut \( C \) that is not tight,
\[
x(C) = \sum_{M \in \mathcal{M}} |M \cap C| \alpha(M) > \sum_{M \in \mathcal{M}} \alpha(M) = 1,
\]
because \( |M \cap C| \geq 1 \) for all perfect matchings \( M \), with strict inequality for at least one perfect matching.

By the first part, the vector \( z := (1 - \epsilon)x + \epsilon y \) is 1-regular for any real number \( \epsilon \). Clearly, for small enough values of \( \epsilon \), the vector \( z \) is also in \( \text{Poly}(G) \).

(iii) Now suppose \( r \neq 1 \), and let \( y \) be an \( r \)-regular vector. Show that \( y \) is in \( \text{Lin}(G) \) by showing that, for any 1-regular vector \( x \), the vector \( \frac{1}{1-r}(x - y) \) is 1-regular.

Exercise 9.5

(i) Let \( T \) denote the matrix whose rows are the incidence vectors of all tight cuts of a matching covered graph \( G \). For any fixed real number \( r \), show that the set of all \( r \)-regular vectors of \( G \) is the set of solutions to the system \( Tx = r \) of linear equations, where \( r \) is a column vector each of whose entries is \( r \).

(Thus, the set of all regular vectors is the set of solutions to the system \( Tx = r \), where \( r \) is treated as a variable).

When \( G \) is either a brace or a brick, all tight cuts are trivial. Thus, in this case, \( T \) is the same as the incidence matrix \( A \) of \( G \). Consequently, \( \text{Lin}(G) \) is the set of solutions to the system \( Ax = r \) of homogeneous linear equations in \( m + 1 \) variables. It should be noted that if \( A \) is the incidence matrix of a matching covered graph \( G \), the sum of columns of \( A \) corresponding to the edges in a perfect matching of \( G \) is \( 1 \). It follows that the dimension of the solution of \( Tx = r \) is \( (m + 1) - \text{rank}(A) \).

(ii) Show that the rank of \( A \) is \( n - 1 \) when \( G \) is bipartite, and is \( n \), when \( G \) non-bipartite, and deduce the validity of the dimension formula for braces and bricks.